Section 2.4

Limits and Continuity

(1) Continuity and Discontinuity

(2) Continuity and Elementary Functions

Continuity can be described as “uninterrupted flow." A function is continuous at a point if the **limit** and **actual value** align at that point.

Continuity at a Point

A function f is **continuous** at the point $(c, f(c))$ if

$$\lim_{x \to c} f(x) = f(c)$$

A few immediate consequences of being continuous are

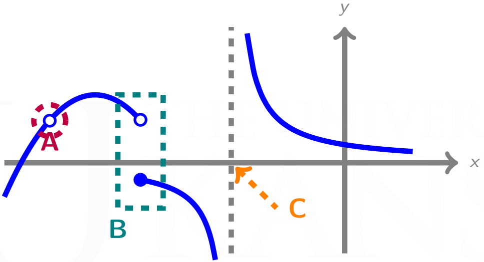
(i) The limit at c, $\lim_{x \to c} f(x)$, exists.

(ii) The actual value, $f(c)$, exists.

(iii) There is a value L where $\lim_{x \to c} f(x) = L = f(c)$.

Discontinuities

If a function is **not** continuous at a point, it is called **discontinuous**.



Three Types of Discontinuities:

- **A** **Removable (Hole):** The limit exists but is not equal to the actual value. See **A** above. $\lim_{x \to c} f(x) \neq f(c)$
- **B** **Jump:** One-sided limits are **not** infinite and the two-sided limit does not exist. See **B** above.
- **C** **Infinite:** Either one-sided limit approaches infinity. See **C** above.

One-Sided Continuity

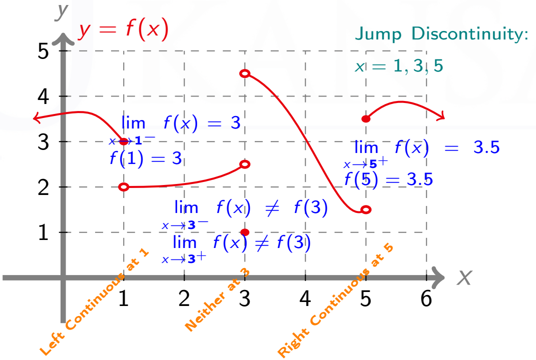
Left Continuity

$$\lim_{x \to c^-} f(x) = f(c)$$

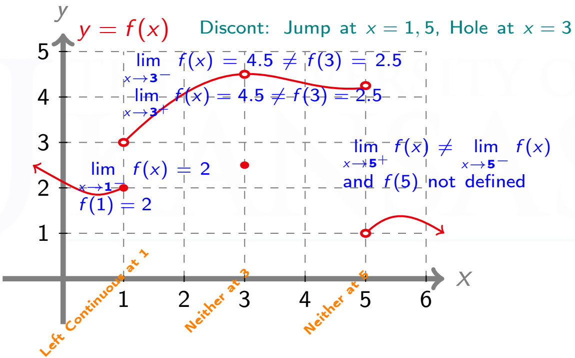
Right Continuity

$$\lim_{x \to c^+} f(x) = f(c)$$

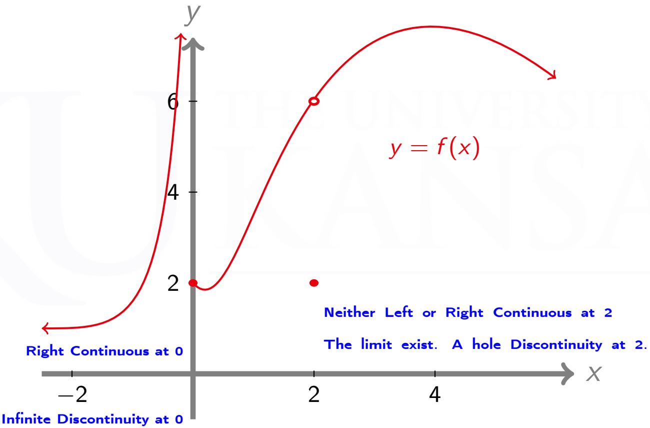
(Example I) Identify any points which are left or right continuous only. Identify and classify any discontinuities.



Continuity and Discontinuity - Example II



Continuity and Discontinuity - Example III



Continuity and Elementary Functions

An **elementary function** is a function of one variable which is the composition of a finite number of operations ($+ \ - \ \times \div$), exponentials, logarithms, constants, polynomials, trigonometric functions, inverse trigonometric functions, and roots.

Elementary Functions are Continuous on Their Domain

For example, $f(x) = \log_5(x^2+3) - e^{\sin(x)}$ and $g(x) = \dfrac{\sqrt{\sin(e^x)}}{2x^4 - x^2 + 1}$.

If $f(x)$ and $g(x)$ are **continuous** at $x = c$, then

- $kf(x)$ is continuous at $x = c$ for a constant k,
- $f(x) \pm g(x)$ is continuous at $x = c$,
- $f(x)g(x)$ is continuous at $x = c$,
- $\frac{f(x)}{g(x)}$ is continuous at $x = c$ provided $g(c) \neq 0$.
- If $f(x)$ is continuous at $x = g(c)$, then $(f \circ g)(x)$ is continuous at $x = c$.

Elementary Functions are Continuous on Their Domain

- Polynomials are continuous at **every** real number.
- **Even** root functions are continuous at **every** <u>positive</u> number.
- **Odd** root functions are continuous at **every** real number.
- Logarithms are continuous at **every** <u>positive</u> number.
- Exponential functions are continuous at **every** real number.
- $\sin(x)$ and $\cos(x)$ are continuous at **every** real number.
- $\tan(x)$ and $\sec(x)$ are continuous everywhere except for values $\frac{\pi}{2} + n\pi$.
- $\cot(x)$ and $\csc(x)$ are continuous everywhere except for values $0 + n\pi$.

Continuity - Example IV

On which intervals are the following functions continuous?

(i) $r(x) = \sqrt{x^2 - 2x - 5}$

Continuous on the the open subset of the domain: $\left(-\infty, \frac{2-\sqrt{24}}{2}\right) \cup \left(\frac{2+\sqrt{24}}{2}, \infty\right)$

(ii) $f(x) = \frac{x^2+\cos(2^x+9)}{x-8}$

Continuous on the domain: $(-\infty, 8) \cup (8, \infty)$

(iii) $h(x) = \frac{3^x}{\sqrt{x+5}}$

Continuous on the domain: $(-5, \infty)$

(iv) $g(x) = e^{\frac{\tan(x)}{x}}$

Continuous on the domain: $x \neq \frac{\pi}{2} + k\pi$, $x \neq 0$

(Example V) Find the value of c that will make the function continuous at $x = 3$.

$$f(x) = \begin{cases} 2x + \frac{9}{x} & \text{for } x \leq 3 \\ -4x + c & \text{for } x > 3 \end{cases}$$ Link

Since $\lim_{x \to 3^-} f(x) = 2(3) + \dfrac{9}{3} = f(3) = 9$ and $\lim_{x \to 3^+} f(x) = -4(3) + c = -12 + c$, for f to be continuous at $x = 3$ $9 = -12 + c$. $\boxed{c = 21}$

(Example VI) Classify the discontinuities of the function.

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & \text{for } x \neq 4 \\ 10 & \text{for } x = 4 \end{cases}$$

$\dfrac{x^2 - 16}{x - 4} = x + 4$ for $x \neq 4$ so f is continuous for $x \neq 4$. Since $\lim_{x \to 4} f(x) = 8 \neq f(4)$, f is discontinuous at $x = 4$. (a hole)
Note: If we replace 10 by 8, f would be continuous everywhere.

Limits and Continuity

$$\text{Continuity: } \lim_{x\to c} f(x) = f(c)$$

When asked to evaluate $\lim_{x\to c} f(x)$, continuity can be extremely useful!

If $f(x)$ is continuous at $x = c$, then the limit **must** be the actual value, $f(c)$; this technique is known as **direct substitution**.

Keep in mind, elementary functions are continuous on their domain!

Composition Limit Law

If f is continuous at $\lim_{x\to c} g(x)$, $\quad \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$

(Example VII) $\lim_{x\to 0} \sqrt{x+1}e^{\tan(x)} = 1e^0 = 1$

Example VIII

$$\lim_{x\to -1} f(x) = 3 \qquad \lim_{x\to -1} g(x) = -2$$

$$\lim_{x\to 2} f(x) = -1 \qquad \lim_{x\to 2} g(x) = 4$$

With the above information, evaluate the limit:

(iii) $$\lim_{x\to -1} \frac{g(-2x)}{x^2} = \frac{\lim_{x\to -1} g(-2x)}{\lim_{x\to -1} x^2} = \frac{\lim_{u\to 2} g(u)}{1} = 4$$

This above is possible because $-2x \neq 2$ near $x = -1$.

(From Section 2.3!)